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Shape invariance, raising and lowering operators in hypergeometric type equations

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Abstract
The Schrödinger equations which are exactly solvable in terms of associated special functions are directly related to some self-adjoint operators defined in the theory of hypergeometric type equations. The fundamental formulae occurring in a supersymmetric approach to these Hamiltonians are consequences of some formulae concerning the general theory of associated special functions. We use this connection in order to obtain a *general theory of Schrödinger equations exactly solvable in terms of associated special functions*, and to extend certain results known in the case of some particular potentials.

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1. Introduction

It is well known [5, 10] that, in the case of certain potentials, the Schrödinger equation is exactly solvable and its solutions can be expressed in terms of the so-called associated special functions. The purpose of this paper is to present a general theory of these quantum systems. Our systematic study recovers a number of earlier results in a natural unified way and also leads to new findings.

The study of the Hamiltonians of these quantum systems can be reduced to the study of some directly related self-adjoint operators defined in the general theory of orthogonal polynomials [23]. For example, in order to factorize a Hamiltonian as a product of two first-order differential operators it is sufficient to factorize the corresponding self-adjoint operator. We show that the self-adjoint operators corresponding to the Hamiltonians of all the considered quantum systems can be studied together in a unified and explicit way. This leads to a general theory of quantum systems exactly solvable in terms of associated special functions.

The number of papers concerning exactly solvable quantum systems and related subjects is very large (see [5, 10, 11, 17, 25] and references therein). Our approach is based on the formalism of the factorization method [10, 22] and on the raising/lowering operators

Table 1. Some important particular cases (the parameters α, β belong to $(-1, \infty)$).

Name	(a, b)	$\sigma(s)$	$\tau(s)$	$\varrho(s)$
Hypergeometric	$(0, 1)$	$s(1-s)$	$(\alpha+1)-(\alpha+\beta+2)s$	$s^\alpha(1-s)^\beta$
Jacobi	$(-1, 1)$	$1-s^2$	$(\beta-\alpha)-(\alpha+\beta+2)s$	$(1-s)^\alpha(1+s)^\beta$
Laguerre	$(0, \infty)$	s	$\alpha+1-s$	$s^\alpha\,\mathrm{e}^{-s}$
Hermite	$(-\infty, \infty)$	1	$-s$	e^{-s^2}

presented in general form (for the first time to our knowledge) by Jafarizadeh and Fakhri [11]. We re-obtain these operators in a simpler way, and use them in a rather different way. In addition, we present some new results and applications concerning these operators.

2. Orthogonal polynomials and associated special functions

Many problems in quantum mechanics and mathematical physics lead to equations of the type

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0 \tag{1}$$

where $\sigma(s)$ and $\tau(s)$ are polynomials of at most second and first degree, respectively, and λ is a constant. These equations are usually called *equations of hypergeometric type*, and the corresponding solutions *functions of hypergeometric type* [23]. Equation (1) can be reduced to the self-adjoint form

$$[\sigma(s)\varrho(s)y'(s)]' + \lambda\varrho(s)y(s) = 0 \tag{2}$$

by choosing a function ϱ such that $[\sigma(s)\varrho(s)]' = \tau(s)\varrho(s)$. For

$$\lambda = \lambda_l = -\tfrac{1}{2}l(l-1)\sigma'' - l\tau' \qquad \text{with} \quad l \in \mathbb{N} \tag{3}$$

there exists a polynomial Φ_l of degree l satisfying (1), that is,

$$\sigma(s)\Phi_l''(s) + \tau(s)\Phi_l'(s) + \lambda_l\Phi_l(s) = 0. \tag{4}$$

If there exists a finite or infinite interval (a, b) such that

$$\sigma(s)\varrho(s)s^k|_{s=a} = 0 \qquad \sigma(s)\varrho(s)s^k|_{s=b} = 0 \qquad \text{for all} \quad k \in \mathbb{N} \tag{5}$$

and if $\sigma(s) > 0, \varrho(s) > 0$ for all $s \in (a, b)$, then the polynomials Φ_l are orthogonal with weight function $\varrho(s)$ in the interval (a, b)

$$\int_a^b \Phi_l(s)\Phi_k(s)\varrho(s)\,\mathrm{d}s = 0 \qquad \text{for} \quad \lambda_l \neq \lambda_k. \tag{6}$$

In this case Φ_l are known as *classical orthogonal polynomials* [23]. We shall prove that the condition $\lambda_l \neq \lambda_k$ from (6) can be replaced by $l \neq k$. The main particular cases of this general approach are presented in table 1.

The classical orthogonal polynomials Φ_l satisfy a three-term recurrence relation

$$s\Phi_l(s) = \alpha_l\Phi_{l+1}(s) + \beta_l\Phi_l(s) + \gamma_l\Phi_{l-1}(s) \tag{7}$$

and Rodrigues formula

$$\Phi_l(s) = \frac{B_l}{\varrho(s)}[\sigma^l(s)\varrho(s)]^{(l)} \tag{8}$$

where $\alpha_l, \beta_l, \gamma_l$ and B_l are constants [23].

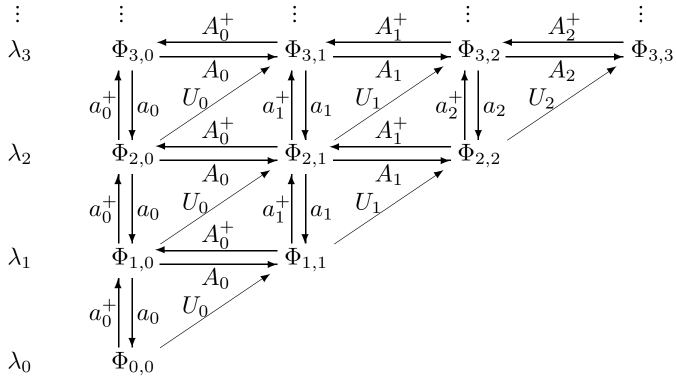

Figure 1. The functions $\Phi_{l,m}$ satisfy the relation $H_m\Phi_{l,m} = \lambda_l\Phi_{l,m}$, and are related (up to some multiplicative constants) through the operators A_m, A_m^+, a_m, a_m^+, U_m and $U_m^{-1} = U_m^+$.

Consider a system of classical orthogonal polynomials, and let $\kappa(s) = \sqrt{\sigma(s)}$. By differentiating equation (4) m times and multiplying it by $\kappa^m(s)$, we get for each $m \in \{0, 1, 2, \ldots, l\}$ the *associated differential equation*

$$-\sigma(s)\Phi''_{l,m} - \tau(s)\Phi'_{l,m} + \left[\frac{m(m-2)}{4}\frac{\sigma'^2(s)}{\sigma(s)} + \frac{m\tau(s)}{2}\frac{\sigma'(s)}{\sigma(s)} - \frac{1}{2}m(m-2)\sigma''(s) - m\tau'(s)\right]\Phi_{l,m} = \lambda_l\Phi_{l,m} \tag{9}$$

where

$$\Phi_{l,m}(s) = \kappa^m(s)\Phi_l^{(m)}(s) \tag{10}$$

are known as the *associated special functions*. We have ([23], p 8)

$$\int_a^b \Phi_{l,m}(s)\Phi_{k,m}(s)\varrho(s)\,\mathrm{d}s = \int_a^b \Phi_l^{(m)}(s)\Phi_k^{(m)}(s)\sigma^m(s)\varrho(s)\,\mathrm{d}s = 0 \tag{11}$$

for any $m \in \mathbb{N}$ and $l, k \in \{m, m+1, m+2, \ldots\}$ with $l \neq k$. This means that for each $m \in \mathbb{N}$, the set $\{\Phi_{m,m}, \Phi_{m+1,m}, \Phi_{m+2,m}, \ldots\}$ (see figure 1) is an orthogonal sequence in the Hilbert space

$$\mathcal{H} = \left\{\varphi : (a,b) \longrightarrow \mathbb{C} \;\middle|\; \int_a^b |\varphi(s)|^2\varrho(s)\,\mathrm{d}s < \infty\right\} \tag{12}$$

with the scalar product given by

$$\langle\varphi, \psi\rangle = \int_a^b \overline{\varphi(s)}\psi(s)\varrho(s)\,\mathrm{d}s. \tag{13}$$

For each $m \in \mathbb{N}$, let $\mathcal{H}_m$ be the linear span of $\{\Phi_{m,m}, \Phi_{m+1,m}, \Phi_{m+2,m}, \ldots\}$. In the following we shall restrict ourself to the case when $\mathcal{H}_m$ is dense in $\mathcal{H}$ for all $m \in \mathbb{N}$. For this it is sufficient that the interval (a, b) is finite, but not necessary.

Equation (9) can be written as

$$H_m\Phi_{l,m} = \lambda_l\Phi_{l,m} \tag{14}$$

where $H_m : \mathcal{H}_m \longrightarrow \mathcal{H}_m$ is the differential operator

$$H_m = -\sigma(s)\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \tau(s)\frac{\mathrm{d}}{\mathrm{d}s} + \frac{m(m-2)}{4}\frac{\sigma'^2(s)}{\sigma(s)} + \frac{m\tau(s)}{2}\frac{\sigma'(s)}{\sigma(s)} - \frac{1}{2}m(m-2)\sigma''(s) - m\tau'(s). \tag{15}$$

The problem of factorization of operators H_m is very important since it is directly related to the factorization of some Schrödinger type operators [5, 10]. If we use in (9) a change of variable $s = s(x)$ such that $\mathrm{d}s/\mathrm{d}x = \kappa(s(x))$ or $\mathrm{d}s/\mathrm{d}x = -\kappa(s(x))$ and define the new functions

$$\Psi_{l,m}(x) = \sqrt{\kappa(s(x))\varrho(s(x))}\,\Phi_{l,m}(s(x)) \tag{16}$$

then we get an equation of the Schrödinger type [10]

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Psi_{l,m}(x) + V_m(x)\Psi_{l,m}(x) = \lambda_l\Psi_{l,m}(x). \tag{17}$$

For example, by starting from the equation of Jacobi polynomials with $\alpha = \mu - 1/2$, $\beta = \eta - 1/2$, and using the change of variable $s(x) = \cos x$ we obtain the Schrödinger equation corresponding to the Pöschl–Teller potential [1, 6]

$$V_0(x) = \frac{1}{4}\left[\frac{\mu(\mu-1)}{\cos^2(x/2)} + \frac{\eta(\eta-1)}{\sin^2(x/2)}\right] - \frac{(\mu+\eta)^2}{4}. \tag{18}$$

3. Raising and lowering operators. Shape invariance

Lorente has shown recently [19, 20] that a factorization of H_0 can be obtained by using the three-term recurrence relation (7) and a consequence of the Rodrigues formula. Following Lorente's idea we obtain a factorization of H_m by using (10) and a three-term recurrence relation.

Theorem 1. *For any $l \in \mathbb{N}$ and any $m \in \{0, 1, \ldots, l-1\}$ we have*

$$\Phi_{l,m+1}(s) = \left(\kappa(s)\frac{\mathrm{d}}{\mathrm{d}s} - m\kappa'(s)\right)\Phi_{l,m}(s). \tag{19}$$

Proof. By differentiating (10) we get

$$\Phi'_{l,m}(s) = m\kappa^{m-1}(s)\kappa'(s)\Phi_l^{(m)}(s) + \kappa^m(s)\Phi_l^{(m+1)}(s)$$

that is, the relation

$$\Phi'_{l,m}(s) = m\frac{\kappa'(s)}{\kappa(s)}\Phi_{l,m}(s) + \frac{1}{\kappa(s)}\Phi_{l,m+1}(s)$$

equivalent to (19). □

Theorem 2. *The three-term recurrence relation*

$$\Phi_{l,m+1}(s) + \left(\frac{\tau(s)}{\kappa(s)} + 2(m-1)\kappa'(s)\right)\Phi_{l,m}(s) + (\lambda_l - \lambda_{m-1})\Phi_{l,m-1}(s) = 0 \tag{20}$$

is satisfied for any $l \in \mathbb{N}$ and any $m \in \{1, 2, \ldots, l-1\}$. In addition, we have

$$\left(\frac{\tau(s)}{\kappa(s)} + 2(l-1)\kappa'(s)\right)\Phi_{l,l}(s) + (\lambda_l - \lambda_{l-1})\Phi_{l,l-1}(s) = 0. \tag{21}$$

Proof. By differentiating (4) $m-1$ times we obtain

$$\sigma(s)\Phi_l^{(m+1)}(s)+(m-1)\sigma'(s)\Phi_l^{(m)}(s)+\frac{(m-1)(m-2)}{2}\sigma''(s)\Phi_l^{(m-1)}(s)$$
$$+\tau(s)\Phi_l^{(m)}+(m-1)\tau'(s)\Phi_l^{(m-1)}(s)+\lambda_l\Phi_l^{(m-1)}(s)=0.$$

If we multiply this relation by $\kappa^{m-1}(s)$ then we get (20) for $m\in\{1,2,\ldots,l-1\}$, and (21) for $m=l$. □

Theorem 3. *For any $l\in\mathbb{N}$ and any $m\in\{0,1,\ldots,l-1\}$ we have the relation*

$$(\lambda_l-\lambda_m)\Phi_{l,m}(s)=\left(-\kappa(s)\frac{\mathrm{d}}{\mathrm{d}s}-\frac{\tau(s)}{\kappa(s)}-(m-1)\kappa'(s)\right)\Phi_{l,m+1}(s). \tag{22}$$

Proof. If $m\in\{1,2,\ldots,l-1\}$ then by substituting (19) into (20) we get

$$\left(\kappa(s)\frac{\mathrm{d}}{\mathrm{d}s}+\frac{\tau(s)}{\kappa(s)}+(m-2)\kappa'(s)\right)\Phi_{l,m}(s)+(\lambda_l-\lambda_{m-1})\Phi_{l,m-1}(s)=0$$

that is, (22) holds for all $m\in\{0,1,\ldots,l-2\}$. In the case $m=l-1$ relation (22) follows directly from (21). □

Theorem 4. *The operators*

$$A_m:\mathcal{H}_m\longrightarrow\mathcal{H}_{m+1}\qquad A_m=\kappa(s)\frac{\mathrm{d}}{\mathrm{d}s}-m\kappa'(s) \tag{23}$$

and

$$A_m^+:\mathcal{H}_{m+1}\longrightarrow\mathcal{H}_m\qquad A_m^+=-\kappa(s)\frac{\mathrm{d}}{\mathrm{d}s}-\frac{\tau(s)}{\kappa(s)}-(m-1)\kappa'(s) \tag{24}$$

are mutually adjoint [24].

Proof. Since $\sigma^m(s)\Phi_l^{(m)}(s)\Phi_k^{(m+1)}(s)$ is a polynomial, from (5) we get

$$\begin{aligned}\langle A_m\Phi_{l,m},\Phi_{k,m+1}\rangle&=\int_a^b[\kappa(s)\Phi_{l,m}'(s)-m\kappa'(s)\Phi_{l,m}(s)]\Phi_{k,m+1}(s)\varrho(s)\,\mathrm{d}s\\&=\kappa(s)\Phi_{l,m}(s)\Phi_{k,m+1}(s)\varrho(s)|_a^b-\int_a^b\Phi_{l,m}(s)[\kappa(s)\Phi_{k,m+1}'(s)\varrho(s)\\&\quad+\kappa(s)\Phi_{k,m+1}(s)\varrho'(s)+(m+1)\kappa'(s)\Phi_{k,m+1}(s)\varrho(s)]\,\mathrm{d}s\\&=\sigma(s)\varrho(s)\sigma^m(s)\Phi_l^{(m)}(s)\Phi_k^{(m+1)}(s)|_a^b+\int_a^b\Phi_{l,m}(s)\left(A_m^+\Phi_{k,m+1}\right)(s)\varrho(s)\,\mathrm{d}s\\&=\left\langle\Phi_{l,m},A_m^+\Phi_{k,m+1}\right\rangle\end{aligned}$$

for any $l\geqslant m$, $k\geqslant m+1$. □

Since

$$\begin{aligned}\|\Phi_{l,m+1}\|^2&=\langle\Phi_{l,m+1},\Phi_{l,m+1}\rangle=\langle A_m\Phi_{l,m},\Phi_{l,m+1}\rangle\\&=\left\langle\Phi_{l,m},A_m^+\Phi_{l,m+1}\right\rangle=(\lambda_l-\lambda_m)\|\Phi_{l,m}\|^2\end{aligned}$$

it follows that $\lambda_l>\lambda_m$ for all $l>m$, and

$$\|\Phi_{l,m+1}\|=\sqrt{\lambda_l-\lambda_m}\,\|\Phi_{l,m}\|. \tag{25}$$

This is possible only if $\sigma''(s)\leqslant 0$ and $\tau'(s)<0$. In particular, we have $\lambda_l\neq\lambda_k$ if and only if $l\neq k$.

Theorem 5. *The operators $H_m : \mathcal{H}_m \longrightarrow \mathcal{H}_m$ are self-adjoint,*

$$H_m - \lambda_m = A_m^+ A_m \qquad H_{m+1} - \lambda_m = A_m A_m^+ \tag{26}$$

and

$$H_m A_m^+ = A_m^+ H_{m+1} \qquad A_m H_m = H_{m+1} A_m \tag{27}$$

for all $m \in \mathbb{N}$.

Proof. Relations (19) and (22) can be written as

$$A_m \Phi_{l,m} = \Phi_{l,m+1} \qquad A_m^+ \Phi_{l,m+1} = (\lambda_l - \lambda_m)\Phi_{l,m} \tag{28}$$

and we get

$$A_m^+ A_m \Phi_{l,m} = (\lambda_l - \lambda_m)\Phi_{l,m} \qquad A_m A_m^+ \Phi_{l,m+1} = (\lambda_l - \lambda_m)\Phi_{l,m+1} \tag{29}$$

that is,

$$\left(A_m^+ A_m + \lambda_m\right)\Phi_{l,m} = \lambda_l \Phi_{l,m} \qquad \left(A_m A_m^+ + \lambda_m\right)\Phi_{l,m+1} = \lambda_l \Phi_{l,m+1}$$

whence

$$H_m = A_m^+ A_m + \lambda_m \qquad H_{m+1} = A_m A_m^+ + \lambda_m.$$

The intertwining relations (27) and the fact that the operators H_m are self-adjoint are direct consequences of (26). □

From (26) we obtain the relation expressing the shape invariance [2, 7, 8] of operators H_m

$$A_m A_m^+ = A_{m+1}^+ A_{m+1} + r_{m+1} \tag{30}$$

where $r_{m+1} = \lambda_{m+1} - \lambda_m = -m\sigma'' - \tau'$. In particular, we have $\lambda_l = \sum_{k=1}^{l} r_k$ and

$$\begin{aligned}
&H_0 = A_0^+ A_0\\
&H_1 = A_0 A_0^+ = A_1^+ A_1 + r_1\\
&H_2 = A_1 A_1^+ + r_1 = A_2^+ A_2 + r_1 + r_2\\
&\ldots\\
&H_{m+1} = A_m A_m^+ + \textstyle\sum_{k=1}^{m} r_k = A_{m+1}^+ A_{m+1} + \sum_{k=1}^{m+1} r_k\\
&\ldots.
\end{aligned} \tag{31}$$

The function $\Phi_{l,l}(s) = \kappa^l(s)\Phi_l^{(l)}(s)$ satisfies the relation $A_l \Phi_{l,l} = 0$, and

$$\Phi_{l,m} = \frac{A_m^+}{\lambda_l - \lambda_m}\frac{A_{m+1}^+}{\lambda_l - \lambda_{m+1}} \cdots \frac{A_{l-2}^+}{\lambda_l - \lambda_{l-2}}\frac{A_{l-1}^+}{\lambda_l - \lambda_{l-1}}\Phi_{l,l} \tag{32}$$

for all $l \in \mathbb{N}$ and $m \in \{0, 1, 2, \ldots, l-1\}$.

The operators A_m and A_m^+ have been previously obtained by Jafarizadeh and Fakhri [11] after a rather long calculation by using the ansatz

$$A_m = f_1(s)\frac{\mathrm{d}}{\mathrm{d}s} + g_1(s) \qquad A_m^+ = f_2(s)\frac{\mathrm{d}}{\mathrm{d}s} + g_2(s). \tag{33}$$

We use this opportunity to correct a minor error existing in [11]. Since Jafarizadeh and Fakhri [11] use for the associated special functions the definition $\Phi_{l,m}(s) = (-1)^m \kappa^m(s)\Phi_l^{(m)}(s)$, the proof of our theorem 1 shows that one has to multiply the expressions of $B_{-(m)}$ and $A_{-(m)}$ from [11] by (-1) in order to get the correct raising/lowering operators. The expression for A_m in the Legendre case has been known for a long time [26].

4. Creation and annihilation operators

For each $m \in \mathbb{N}$, the sequence $\{|m, m\rangle, |m+1, m\rangle, |m+2, m\rangle, \ldots\}$, where

$$|l, m\rangle = \Phi_{l,m}/\|\Phi_{l,m}\| \tag{34}$$

is an orthonormal basis of $\mathcal{H}$, and

$$U_m : \mathcal{H} \longrightarrow \mathcal{H} \qquad U_m|l, m\rangle = |l+1, m+1\rangle \tag{35}$$

is a unitary operator.

Theorem 6. *The operators (see figure 1)*

$$a_m, a_m^+ : \mathcal{H}_m \longrightarrow \mathcal{H}_m \qquad a_m = U_m^+ A_m \qquad a_m^+ = A_m^+ U_m \tag{36}$$

are mutually adjoint, and

$$\begin{aligned} a_m|l, m\rangle &= \sqrt{\lambda_l - \lambda_m}|l-1, m\rangle \qquad &&\textit{for all} \quad l \geqslant m+1 \\ a_m^+|l, m\rangle &= \sqrt{\lambda_{l+1} - \lambda_m}|l+1, m\rangle \qquad &&\textit{for all} \quad l \geqslant m. \end{aligned} \tag{37}$$

Proof. This result follows from (25) and the fact that A_m and A_m^+ are mutually adjoint. □

For each $l > m$ we have

$$|l, m\rangle = \frac{\left(a_m^+\right)^{l-m}}{\sqrt{(\lambda_l - \lambda_m)(\lambda_{l-1} - \lambda_m)\cdots(\lambda_{m+1} - \lambda_m)}}|m, m\rangle. \tag{38}$$

Since

$$a_m a_m^+ \Phi_{l,m} = (\lambda_{l+1} - \lambda_m)\Phi_{l,m} \qquad a_m^+ a_m \Phi_{l+1,m} = (\lambda_{l+1} - \lambda_m)\Phi_{l+1,m} \tag{39}$$

we get the factorization

$$H_m - \lambda_m = a_m^+ a_m \tag{40}$$

and the relation

$$\left[a_m, a_m^+\right]\Phi_{l,m} = (\lambda_{l+1} - \lambda_l)\Phi_{l,m}. \tag{41}$$

By using the operator $R_m = -\sigma'' N_m - \tau'$, where N_m is the number operator

$$N_m : \mathcal{H}_m \longrightarrow \mathcal{H}_m \qquad N_m \Phi_{l,m} = l\Phi_{l,m} \tag{42}$$

relation (41) can be written as [2, 6]

$$\left[a_m, a_m^+\right] = R_m. \tag{43}$$

Since

$$\left[a_m^+, R_m\right] = \sigma'' a_m^+ \qquad [a_m, R_m] = -\sigma'' a_m \tag{44}$$

it follows that the Lie algebra $\mathcal{L}_m$ generated by $\left\{a_m^+, a_m\right\}$ is finite dimensional.

Theorem 7.

$$\mathcal{L}_m \textit{ is isomorphic to } \begin{cases} su(1,1) & \textit{if} \quad \sigma'' < 0 \\ \textit{Heisenberg–Weyl algebra} & \textit{if} \quad \sigma'' = 0. \end{cases} \tag{45}$$

Proof. If $\sigma'' < 0$ then $K_+ = \sqrt{-2/\sigma''}a_m^+$, $K_- = \sqrt{-2/\sigma''}a_m$ and $K_0 = (-1/\sigma'')R_m$ form a basis of $\mathcal{L}_m$ such that

$$[K_0, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = -2K_0.$$

In the case $\sigma'' = 0$ the operator R_m is a constant operator, namely, $R_m = -\tau'$. Since $\tau' < 0$, the operators $P_+ = \sqrt{-1/\tau'}a_m^+$, $P_- = \sqrt{-1/\tau'}a_m$ and the identity operator I form a basis of $\mathcal{L}_m$ such that

$$[P_+, P_-] = -I \qquad [P_+, I] = 0 \qquad [P_-, I] = 0$$

that is, $\mathcal{L}_m$ is isomorphic to the Heisenberg–Weyl algebra $h(2)$. $\square$

The result presented in theorem 7 is a direct consequence of the fact that λ_l is a polynomial function of l of at most second degree [2, 25]. Since in the case of the quantum systems considered in this paper we cannot have $\sigma'' > 0$, the Lie algebra $\mathcal{L}_m$ cannot be isomorphic to $su(2)$.

By using (36), relation (43) can be written as

$$U_m^+ A_m A_m^+ U_m - A_m^+ U_m U_m^+ A_m = R_m$$

and in view of (61) we get

$$U_m^+ (H_{m+1} - \lambda_m) U_m - (H_m - \lambda_m) = R_m$$

that is, the relation expressing the shape invariance [8] of H_m

$$H_{m+1} = U_m (H_m + R_m) U_m^+. \tag{46}$$

One can also remark that

$$A_m R_m = R_{m+1} A_m \qquad R_m A_m^+ = A_m^+ R_{m+1} \tag{47}$$

$$[H_m, a_m] = -R_m a_m \qquad \left[H_m, a_m^+\right] = a_m^+ R_m \tag{48}$$

and

$$U_m R_m U_m^+ = R_{m+1} + \sigma'' \tag{49}$$

for all $m \in \mathbb{N}$.

The role played by the Lie algebra $su(1, 1)$ in the case of our quantum systems is more important. In certain cases, Schrödinger equation can be transformed into the differential equation of orthogonal polynomials in group-theoretical terms [17].

5. Systems of coherent states

Let $m \in \mathbb{N}$ be a fixed natural number, and let

$$|n\rangle = |m+n, m\rangle \qquad e_n = \lambda_{m+n} - \lambda_m \tag{50}$$

for all $n \in \mathbb{N}$. Since

$$0 = e_0 < e_1 < e_2 < \cdots < e_n < \cdots \tag{51}$$

and

$$a_m|n\rangle = \sqrt{e_n}|n-1\rangle \qquad a_m^+|n\rangle = \sqrt{e_{n+1}}|n+1\rangle \qquad (H_m - \lambda_m)|n\rangle = e_n|n\rangle \tag{52}$$

we can define a system of coherent states by using the general setting presented in [1].

Let

$$\varepsilon_n = \begin{cases} 1 & \text{if} \quad n = 0 \\ e_1 e_2 \ldots e_n & \text{if} \quad n > 0. \end{cases} \tag{53}$$

If

$$R = \limsup_{n\to\infty} \sqrt[n]{\varepsilon_n} \neq 0 \tag{54}$$

then we can define

$$|z\rangle = \frac{1}{N(|z|^2)} \sum_{n\geqslant 0} \frac{z^n}{\sqrt{\varepsilon_n}} |n\rangle \qquad \text{where} \quad (N(|z|^2))^2 = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\varepsilon_n} \tag{55}$$

for any z in the open disc $C(0, R)$ of centre 0 and radius R. In this way, we get a continuous family $\{|z\rangle | z \in C(0, R)\}$ of normalized coherent states, eigenstates of the operator a_m

$$a_m|z\rangle = z|z\rangle. \tag{56}$$

6. Application to Schrödinger type operators

We have already seen that the operators H_m are directly related to some Schrödinger type operators. If we use a change of variable $s = s(x)$ such that $\mathrm{d}s/\mathrm{d}x = \kappa(s(x))$, then the operators corresponding to A_m and A_m^+ are the adjoint conjugate operators

$$\begin{aligned} \mathcal{A}_m &= [\kappa(s)\varrho(s)]^{1/2} A_m [\kappa(s)\varrho(s)]^{-1/2}|_{s=s(x)} = \frac{\mathrm{d}}{\mathrm{d}x} + W_m(x) \\ \mathcal{A}_m^+ &= [\kappa(s)\varrho(s)]^{1/2} A_m^+ [\kappa(s)\varrho(s)]^{-1/2}|_{s=s(x)} = -\frac{\mathrm{d}}{\mathrm{d}x} + W_m(x) \end{aligned} \tag{57}$$

where the *superpotential* $W_m(x)$ is given by the formula

$$W_m(x) = -\frac{\tau(s(x))}{2\kappa(s(x))} - \frac{2m-1}{2\kappa(s(x))}\frac{\mathrm{d}}{\mathrm{d}x}\kappa(s(x)). \tag{58}$$

From (28) and (29) we get

$$\mathcal{A}_m \Psi_{l,m}(x) = \Psi_{l,m+1}(x) \qquad \mathcal{A}_m^+ \Psi_{l,m+1}(x) = (\lambda_l - \lambda_m)\Psi_{l,m}(x) \tag{59}$$

and

$$\left(\mathcal{A}_m^+ \mathcal{A}_m + \lambda_m\right) \Psi_{l,m} = \lambda_l \Psi_{l,m} \qquad \left(\mathcal{A}_m \mathcal{A}_m^+ + \lambda_m\right) \Psi_{l,m+1} = \lambda_l \Psi_{l,m+1} \tag{60}$$

whence

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_m(x) - \lambda_m = \mathcal{A}_m^+ \mathcal{A}_m \qquad -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_{m+1}(x) - \lambda_m = \mathcal{A}_m \mathcal{A}_m^+ \tag{61}$$

and

$$V_m(x) - \lambda_m = W_m^2(x) - \dot{W}_m(x) \qquad V_{m+1}(x) - \lambda_m = W_m^2(x) + \dot{W}_m(x) \tag{62}$$

where the dot means derivative with respect to x.

Since $\mathcal{A}_m \Psi_{m,m} = 0$, from (57) and (61) we get

$$\dot{\Psi}_{m,m} + W_m(x)\Psi_{m,m} = 0 \qquad -\ddot{\Psi}_{m,m} + (V_m(x) - \lambda_m)\Psi_{m,m} = 0 \tag{63}$$

whence

$$W_m(x) = -\frac{\dot{\Psi}_{m,m}(x)}{\Psi_{m,m}(x)} \qquad V_m(x) = \frac{\ddot{\Psi}_{m,m}(x)}{\Psi_{m,m}(x)} + \lambda_m. \tag{64}$$

For each $m \in \{0, 1, 2, \ldots, l-1\}$ we have

$$\Psi_{l,m}(x) = \frac{\mathcal{A}_m^+}{\lambda_l - \lambda_m} \frac{\mathcal{A}_{m+1}^+}{\lambda_l - \lambda_{m+1}} \cdots \frac{\mathcal{A}_{l-2}^+}{\lambda_l - \lambda_{l-2}} \frac{\mathcal{A}_{l-1}^+}{\lambda_l - \lambda_{l-1}} \Psi_{l,l}(x). \tag{65}$$

If we choose the change of variable $s = s(x)$ such that $\mathrm{d}s/\mathrm{d}x = -\kappa(s(x))$, then formulae (57), (58), (62) and (64) become

$$\mathcal{A}_m = -\frac{\mathrm{d}}{\mathrm{d}x} + W_m(x) \qquad \mathcal{A}_m^+ = \frac{\mathrm{d}}{\mathrm{d}x} + W_m(x) \tag{66}$$

$$W_m(x) = -\frac{\tau(s(x))}{2\kappa(s(x))} + \frac{2m-1}{2\kappa(s(x))}\frac{\mathrm{d}}{\mathrm{d}x}\kappa(s(x)) \tag{67}$$

$$V_m(x) - \lambda_m = W_m^2(x) + \dot{W}_m(x) \qquad V_{m+1}(x) - \lambda_m = W_m^2(x) - \dot{W}_m(x) \tag{68}$$

$$W_m(x) = \frac{\dot{\Psi}_{m,m}(x)}{\Psi_{m,m}(x)} \qquad V_m(x) = \frac{\ddot{\Psi}_{m,m}(x)}{\Psi_{m,m}(x)} + \lambda_m \tag{69}$$

respectively. For example, in the case of Pöschl–Teller potential (18) we obtain

$$W_0(x) = \frac{1}{2}\left[\mu \cot\frac{x}{2} - \eta \tan\frac{x}{2}\right]. \tag{70}$$

7. Concluding remarks

The Schrödinger equations which are exactly solvable in terms of associated special functions are directly related to the self-adjoint operators H_m defined by (15), and hence, each of them can be described by the interval (a, b) and the corresponding functions $\sigma(s), \tau(s), \varrho(s), s(x)$ satisfying the conditions presented in section 2. This infinite class of exactly solvable problems depending on several parameters contains well-known potentials (together with their supersymmetric partners) as well as other physically relevant potentials [11].

Our results concerning the operators H_m allow these quantum systems to be studied together in a unitary way, and to extend certain results presented up to now only in the case of some particular potentials. In order to pass to a particular potential it suffices to replace $\sigma(s), \tau(s), \varrho(s), s(x)$ by the corresponding functions. The results concerning the creation/annihilation operators and the coherent states are direct extensions of some results presented in [1, 6] in the particular case of Pöschl–Teller potentials.

In this paper we analyse an important class of exactly solvable Schrödinger equations, but the class of known solvable problems is larger [3, 4, 12, 13, 18]. Generally, the methods used to enlarge the class of exactly solvable potentials are based on the idea of finding pairs of (almost) isospectral operators, and the construction of new exactly solvable Hamiltonians starts from a known exactly solvable Hamiltonian. In most of the cases the starting Hamiltonian belongs to the class considered in this paper.

We have already seen that the superpotential W_m which allows the construction of the supersymmetric partner V_{m+1} of V_m satisfies the Riccati equation

$$V_m(x) - \lambda_m = W_m^2(x) - \dot{W}_m(x). \tag{71}$$

New supersymmetric partners of V_m can be obtained by finding new solutions W of this equation [12, 13] or by solving the more general Riccati equation [3, 21]

$$V_m(x) - \varepsilon = W^2(x) - \dot{W}(x). \tag{72}$$

The usual algebraic approach can also be extended to these new exactly solvable potentials by using some nonlinear generalizations of Lie algebras [4]. Certain algebraic properties become more transparent if we use the method proposed recently by Gurappa *et al* [9] which allows the space of the solutions of a linear differential equation to be connected to the space of monomials, but the advantages obtained in the case of our Hamiltonians are not very significant.

The one-dimensional problems considered in this paper are useful for solving separable potential problems. It is known [5] that a problem is algebraically solvable so long as the separated problems of each of the coordinates can be solved algebraically. In the case of certain potentials, the Schrödinger equation admits separation of variables in two or more coordinate systems [14–16].

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